By equating the strain energy in Figs. 1 and 2 the stiffness matrix defined by the relation, $F = ku$, is obtained

$$[k] = \beta\, \alpha^{-1} \beta^T \tag{6}$$

$$[k] = \frac{12}{a^2 + 12g}\,\frac{EI}{a}\begin{bmatrix} 1 & & & \text{sym} \\ a/2 & a^2/3 + g & & \\ -1 & -a/2 & 1 & \\ a/2 & a^2/6 - g & -a/2 & a^2/3 + g \end{bmatrix} \tag{7}$$

The stiffness matrix given in Eq. (7) is exactly the same as the one obtained in Ref. 1 using a displacement formulation.

The important conclusion from the derivation given in this comment is that by using the flexibility matrix one can include the shear deformation without worrying about the differences between the first derivative of the transverse displacement and the rotation. These differences are distinguished automatically.

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[1]Naravanaswami, R. and Adelman, H.M., "Inclusion of Transverse Shear Deformation in Finite Element Displacement Formulations," *AIAA Journal*, Vol. 12, Nov. 1974, pp. 1613-1614.

Reply by Authors to M. Baruch

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M. BARUCH has correctly pointed out that inclusion of transverse shear deformation in finite element force (flexibility) formulations does not require choosing between the first derivatives of the transverse displacements and the rotations of the normals to the neutral surface of the finite element as degrees of freedom. Once the flexibility matrix has been derived in terms of an appropriate set of forces and moments and inverted to form the stiffness matrix, the generalized coordinates associated with the rows and columns of the stiffness matrix are guaranteed to be the correct variables. Incidently, the flexibility matrix derived by Baruch may also be found in Ref. 1.

The point of Ref. 2 as stated in the conclusions was that, in contrast to previous assertions,[3,4] the finite element *displacement* approach proceeds by a straight-forward energy minimization to yield the correct element stiffness matrix even when transverse shear deformation is included. To obtain the correct result by the displacement approach, however, one must use the correct rotational degrees of freedom, namely the rotations of the normals to the middle surface.

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Comment on "Stability of a Spinning Satellite with Flexible Antennas"

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A RECENT paper by Dong and Schlack[1] presents a stability analysis of a spinning flexible satellite via the assumed modes method, whereby the displacements of the flexible parts are represented by series of space-dependent admissible functions multiplied by time-dependent generalized coordinates. As admissible functions they use the fixed-base cantilever modes. Neither the procedure nor the results are new, however.

The problem considered in Ref. 1 was first investigated by Meirovitch and Nelson,[2] who used the assumed modes method in conjunction with an infinitesimal analysis to test the stability of precisely the same mathematical model as that of Ref. 1. In fact, the parameter plot of Ref. 1 (Fig. 2) was originally presented in Ref. 2 (Fig. 2). The stability analysis of both a torque-free spinning flexible satellite and a gravity-gradient stabilized flexible satellite was performed by Meirovitch and Calico[3,4] via the Liapunov direct method in conjunction with the assumed modes method (in addition to the method of integral coordinates and the method of testing density functions). Although Ref. 3 is concerned with a more complicated mathematical model than that of Ref. 2 (and hence than that of Ref. 1), in the sense that the system contains four radial booms in addition to the two axial booms, it does consider also the mathematical model of Ref. 2 for comparison purposes.

The authors of Ref. 1 imply that the ability to derive closed-form stability criteria in terms of infinite series represents a new development. A careful examination of both Refs. 3 and 4, however, reveals that the closed-form stability criteria derived in these references by the assumed modes method are indeed in terms of infinite series. In fact, they correspond exactly to those of Ref. 1, as for the assumed modes method the bounding properties of Rayleigh's quotient need not be used and were in fact not used. The Hessian matrix tested for sign definiteness can be shown[5] to have the form

$$[H]_E = [I + b_i \delta_{ij}]$$

where $b_i (i = 0,1,2,...)$ are real numbers depending on the system parameters. Using Sylvester's criterion, and con-

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sidering the fact that $b_0 < 0$, the following stability criteria are obtained in the general form

$$\sum_{i=0}^{p} \frac{1}{b_i} + 1 < 0, \qquad p = 0, 1 \ldots, n$$

where n is an integer related to the number of admissible functions used in the series representing the elastic displacements. Note that the stability criteria derived in Ref. 3 are more general than those of Ref. 1, and reduce to those of Ref. 1 if the radial booms are omitted.

Dong and Schlack seem to attach considerable importance to infinite series as opposed to finite series. Whether a series is infinite or finite should not be overemphasized, however, as it is seldom necessary to take a large number of terms in the series. Indeed, for the mathematical model in question, Ref. 2 shows that the inclusion of a second mode in the series have virtually no effect on the stability regions, a fact also confirmed by Ref. 1. It is safe to say that in general an infinite series is never summed up for the purpose of producing stability criteria by the assumed modes method, and practical computational considerations demand series truncation at some point.

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Reply by Author to R. A. Calico and L. Meirovitch

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UNFORTUNATELY, W. N. Dong is not available to participate in this reply to the Technical Comment. Consequently, I will attempt to represent our viewpoints on the issues raised in order to submit this Reply in time for it to appear in the same issue as the Comment.

The authors of the Technical Note[1] under discussion made every attempt to give proper reference to the related publications of Meirovitch and Calico,[2,3] since these are important papers and we share the high professional regard that exists for their numerous contributions to the elastic satellite dynamics literature. In the preparation of Ref. 1, we were aware of numerous papers which involve models of satellites with flexible antennas used for comparing various analytical techniques, including those referred to in the second paragraph of the Technical Comment above. The capability of comparing results with other published research is precisely the reason that a satellite model with flexible antennas was selected for study in Ref. 1.

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Although the satellite model in Ref. 1 involves axial antennas only, the motion of the mass center of the satellite due to elastic flexibility is included in the analysis in order to derive infinite series solutions that account for these effects. In the examples discussed in Refs. 2 and 3, the motion of the mass center was assumed negligible. Consequently, the mathematical model as well as the method of analysis in Ref. 1 are sufficiently different from Refs. 2 and 3 that the proposed notion of Ref. 1 being simply a reduced case of these earlier studies is unfounded. The stable regions presented in Fig. 2 of Ref. 1 are somewhat larger than those of Ref. 2 and they are in close agreement with the stability diagrams presented by Kulla,[4] whose work is in agreement with various studies using different analytical techniques. The erroneous implication in Ref. 1 that the Rayleigh quotient approximation was used throughout Refs. 2 and 3 was unintentional.

The statement made that "Ref. 1 implies that the ability to derive closed-form stability criteria in terms of infinite series represents a new development" is also unfounded. Quite the contrary, since clear reference is made in Ref. 1 to a general procedure for developing closed-form solutions for stability criteria in terms of infinite series by Brown and Schlack,[6] as early as August 1969. The statement made in Ref. 1 that "most investigators have found it necessary to truncate their assumed modal series before introducing it into the analysis for determining stability criteria" does not exclude the existence of other series solutions such as those presented in Refs. 2 and 3, for example.

Regarding the importance of infinite series solutions, it is well known that theoretical questions and practical considerations related to series truncation have important implications in both research and design. It is now becoming increasingly clear that useful stability criteria can often be derived by using only a very few series terms. However, several authors have discussed fundamental questions regarding modal series truncation error, including an interesting discussion by Meirovitch and Calico in Ref. 3 suggesting a need for increased research attention in this area.

References

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